Neuronal Models - part IV

Networks

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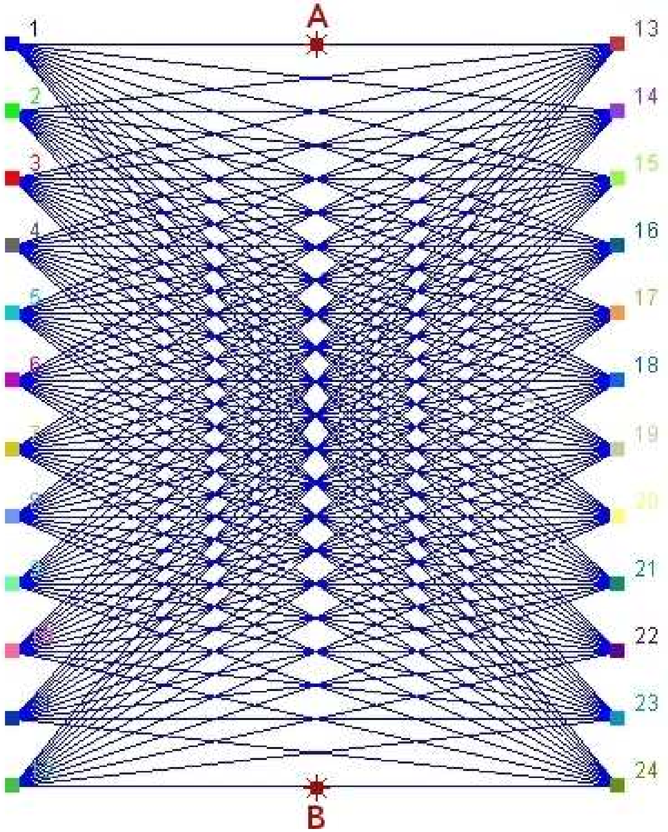
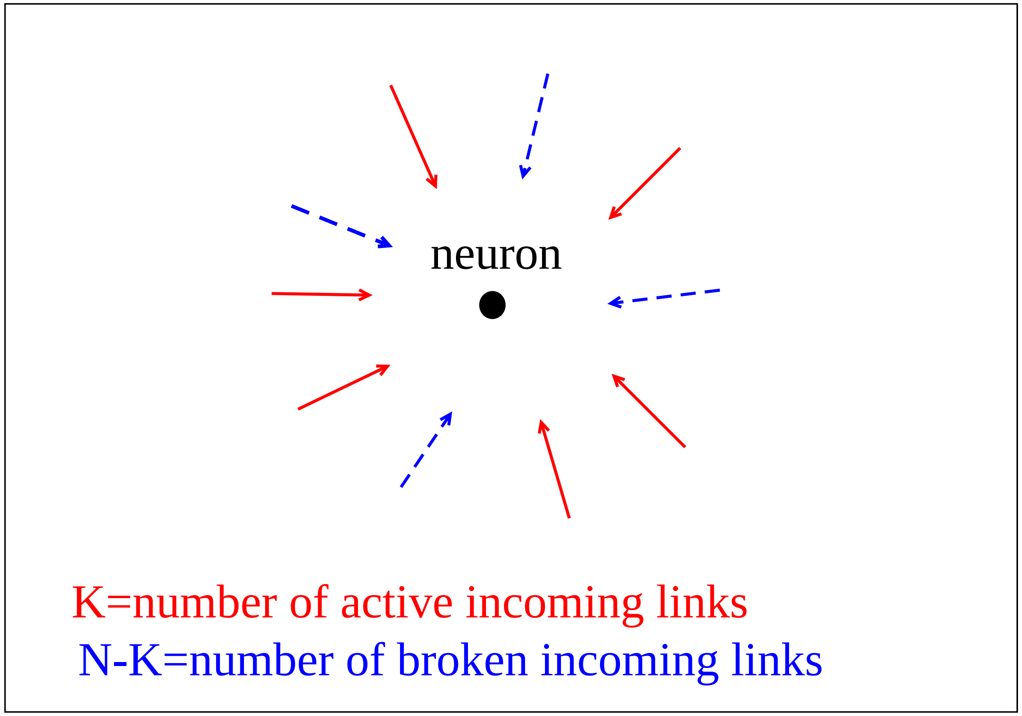
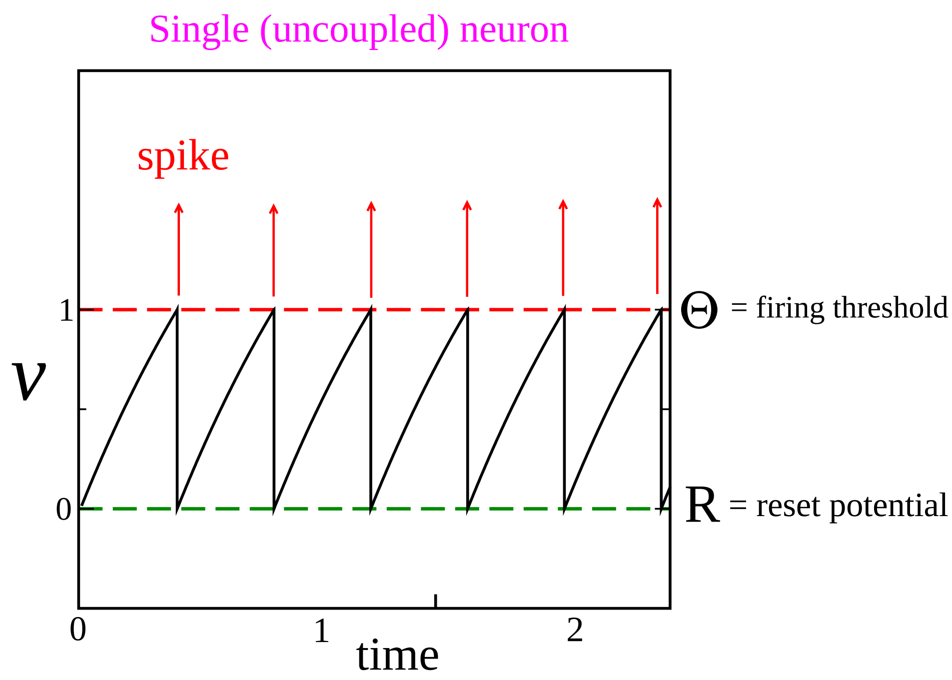
Setups for connectivity





- links are broken randomly and asymmetrically $\rightarrow$ quenched disorder
- two setups for the connectivity K:
  - massively connected networks $\rightarrow K \propto N$ ($K = N$ for fully coupled)
  - sparse networks $\rightarrow K = cost$ for $N \rightarrow \infty$

Networks of pulse-coupled L.I.F.



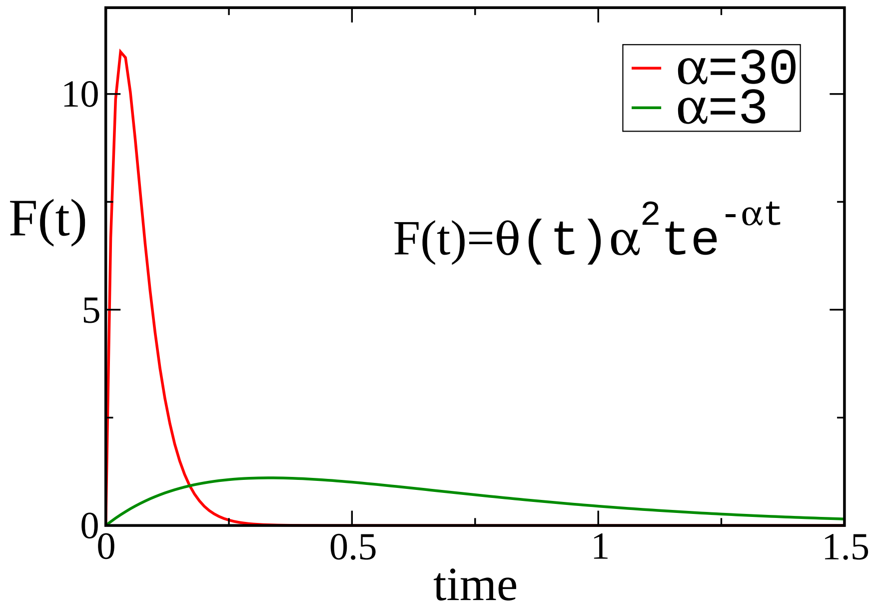
$$\dot{v} = a - v \qquad t < t_f \qquad \rightarrow \qquad v(t) = a(1 - \exp(-\mathrm{t}))$$

$$v(t_f^-) = \Theta \qquad \rightarrow \qquad \text{spike emission}$$

$$v(t_f^+) = R$$

- a=external current $> \Theta \rightarrow$ periodic firing (see the picture)
- a=external current $< \Theta \rightarrow$ silent neuron (no spikes!)

Networks of pulse-coupled L.I.F.



Dynamics of a network of N pulse-coupled LIF [a]:

$$\dot{v}_i = a - v_i + \frac{g}{N} \sum_{n|t_n<t} S_{i,l(n)} F(t - t_n - \tau) \qquad\qquad i = 1, N$$

- $g > 0$ excitatory, $g < 0$ inhibitory coupling strenght
- $S_{ij} \longrightarrow$ connectivity matrix
- $F(t) \longrightarrow$ pulse shape (with unit area)
- $\tau \longrightarrow$ synaptic delay ($\tau = 0$ if not otherwise speficied)

[a] L.F. Abbott and C. van Vreeswijk, Phys. Rev. E 48, 1483 (1993)

An example: α-pulse

α-pulse: $F(t) = \theta(t)\alpha^2 t \mathrm{e}^{-\alpha t}$

(1) $$\dot{v}_i = a - v_i + gE_i \qquad i = 1, N$$

(2) $$\ddot{E}_i + 2\alpha\dot{E}_i + \alpha^2 E_i = \frac{\alpha^2}{N} \sum_{n|t_n<t} S_{i,l(n)}\delta(t - t_n) \qquad i = 1, N$$

- the differential equation for E_i is of second order ($d = 2$) $\rightarrow$ we need two variables for each neuron (the field E_i and its first time derivative $\dot{E}_i$)
- equations simplify using $\{E_i, Q_i\}$ with $Q_i \equiv \alpha E_i + \dot{E}_i$ $\rightarrow$ Eq. 2 can be rewritten in:

(3) $$\dot{E}_i + \alpha E_i = Q_i \qquad i = 1, N$$

(4) $$\dot{Q}_i + \alpha Q_i = \frac{\alpha^2}{N} \sum_{n|t_n<t} S_{i,l(n)}\delta(t - t_n) \qquad i = 1, N$$

An example: α-pulse

"EVENT DRIVEN MAP"

By integrating Eqs.1,3,4 in the time interval between two consecutive spikes of the network (at time t_n and at time t_{n+1}) we get an evolution map for the variables $\{v_i, E_i, Q_i\}$: [a]

$$v_i(n+1) = v_i(n)\mathrm{e}^{-\Delta t} + a(1 - e^{-\Delta t}) + gH[\Delta t, E_i(n), Q_i(n)]$$

$$E_i(n+1) = E_i(n)\mathrm{e}^{-\alpha\Delta t} + Q_i(n)\Delta t\mathrm{e}^{-\alpha\Delta t}$$

$$Q_i(n+1) = Q_i(n)e^{-\alpha\Delta t} + S_{i,l}\frac{\alpha^2}{N} \qquad \Leftarrow \qquad \text{effect of spike from l (firing neuron) to i}$$

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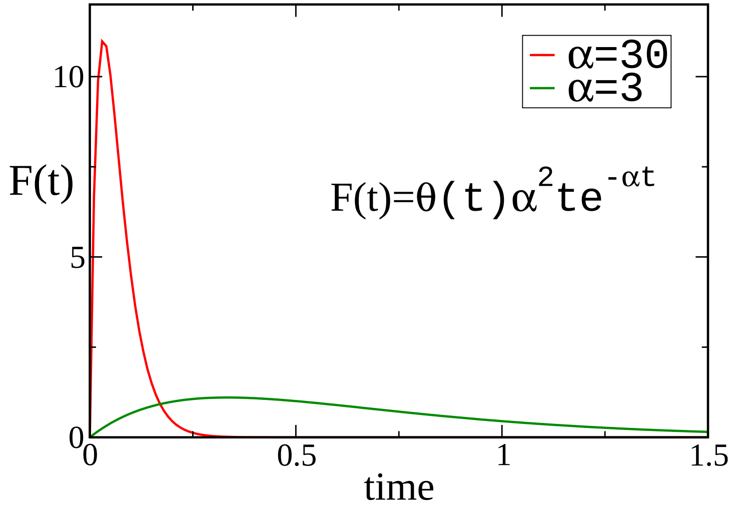
[a] where:

$$H[\Delta t, E, Q] = \frac{\mathrm{e}^{-\Delta t} - e^{-\alpha\Delta t}}{\alpha - 1}\left(E + \frac{Q}{\alpha - 1}\right) - \frac{\Delta t e^{-\alpha\Delta t}}{(\alpha - 1)}Q$$

$$\Delta t = \min\{(\Delta t)_i : v_i(n+1) = 1\} \qquad i \in \{1, N\}$$

Fully coupled network



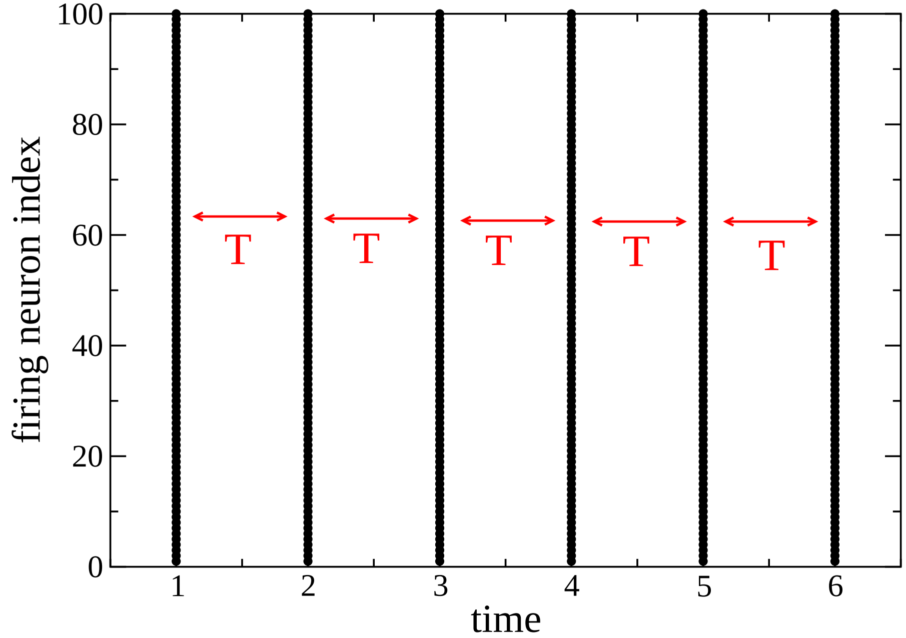
For fully coupled networks:

- $E_i(t) = E(t) \quad \forall i = 1, ..., N$
- the microscopic dynamics is periodic (Splay State, Fully Synchronized, Synchronized Clusters) or quasi-periodic (Partial Synchronization)

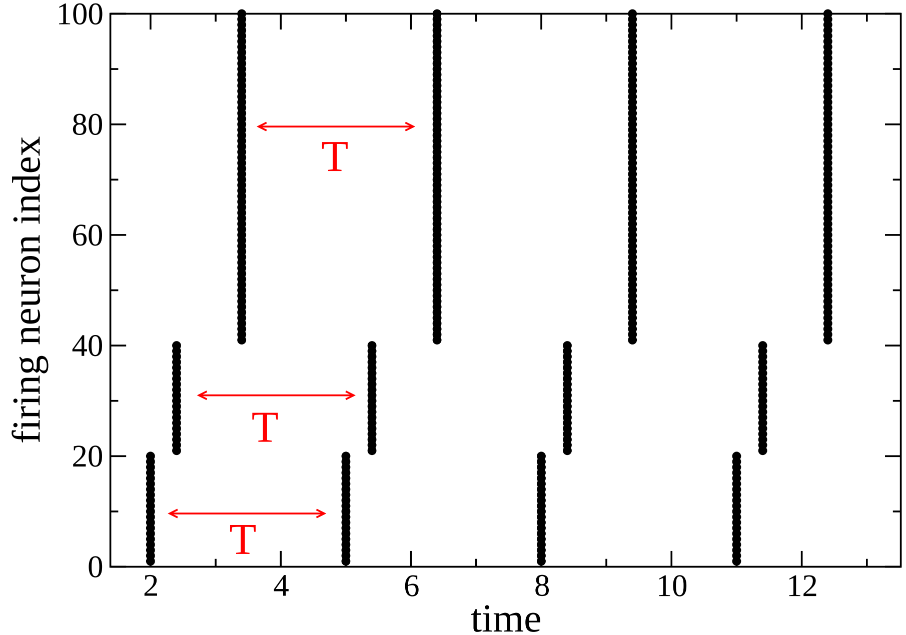
- Excitatory Coupling - $g > 0$
  - Low α – Splay State
  - Larger α – Partially Synchronized State
  - $\alpha \to \infty$ (δ-spike) – Fully Synchronized State
- Inhibitory Coupling - $g < 0$
  - Low α – Fully Synchronized State
  - Larger α – Synchronized Clusters (also one cluster) $\to$ average number of clusters increases with α (number of clusters depends on initial conditions)
  - $\alpha \to \infty$ (δ-spike) – Splay State

Synchronized Clusters

synchronous (1-cluster) state

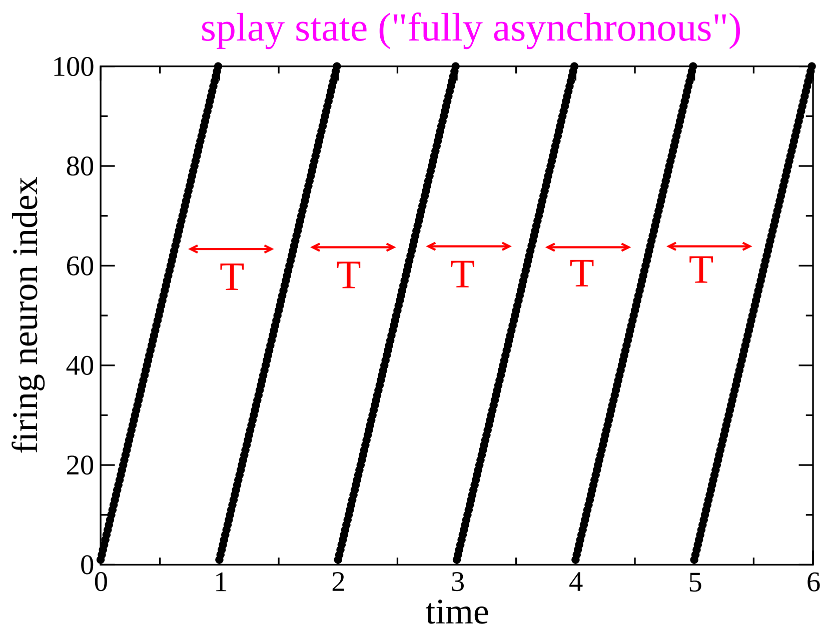


3-clusters state



Splay State

Splay States are collective solutions emerging in Homogeneous Networks of N neurons

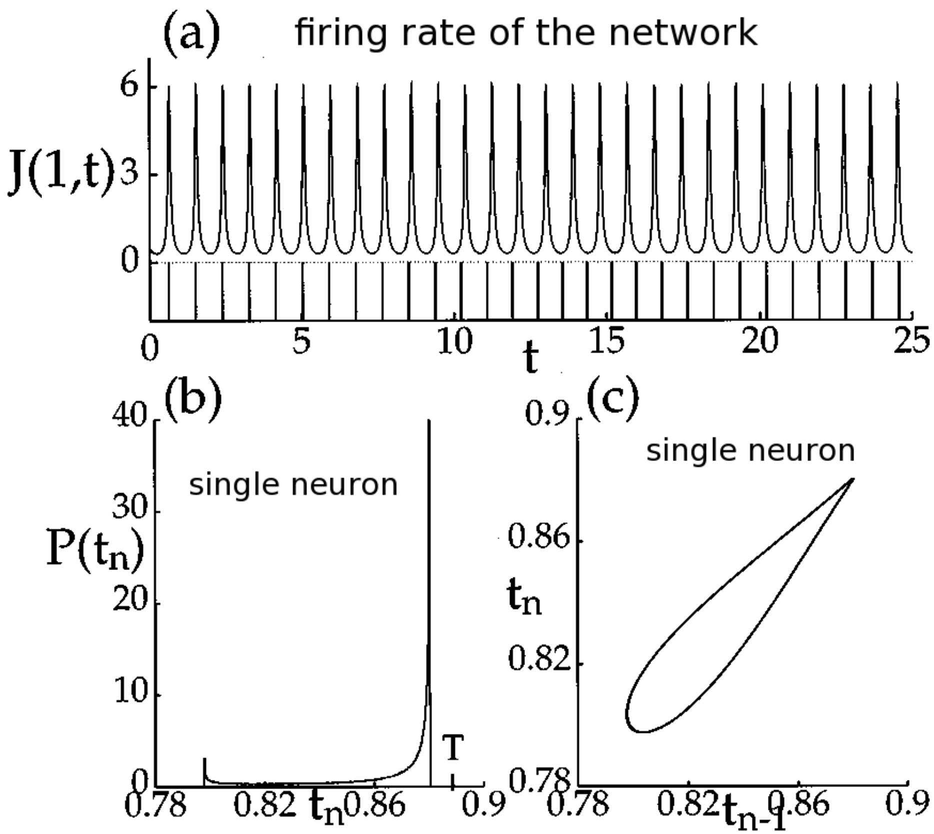


- $t_{n+1} - t_n = T/N = cost$, where t_n and t_{n+1} are consecutive spikes of the network and T is the period of each neuron
- the microscopic dynamics of each neuron is periodic
- the macroscopic dynamics of the network is Asynchronous $\longrightarrow$ the network firing rate N/T and the field $E(t)$ are constant

Abbott - van Vreeswijk, PRE (1993) -- Zillmer et al. PRE (2007)

Partial Synchronization - 1

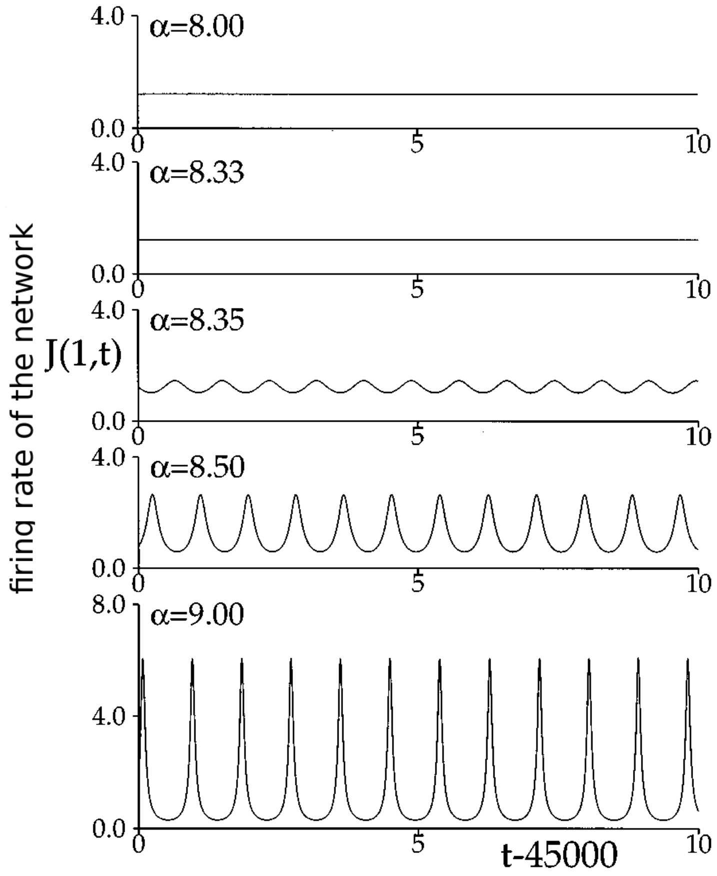
Partial Synchronization in excitatory fully coupled networks (Phys. Rev. E 54, 5522 (1996))

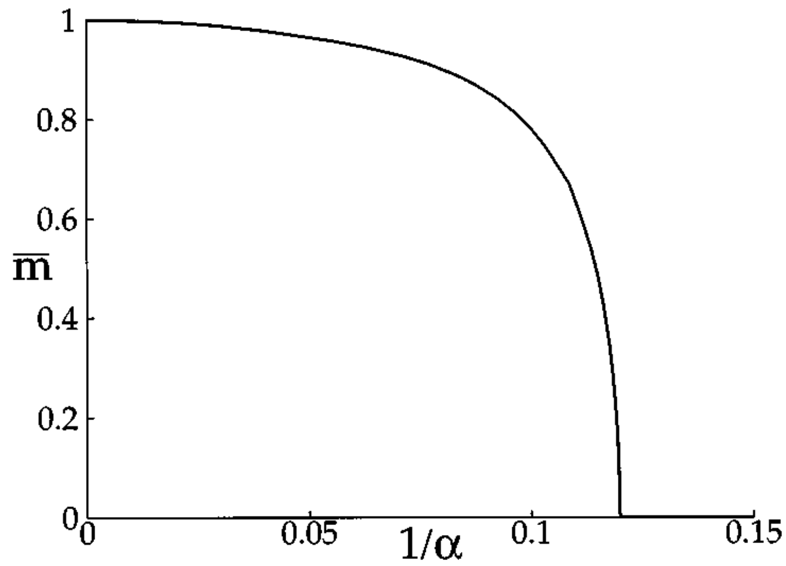


- the microscopic dynamics of single neuron is quasi periodic
- the macroscopic dynamics (firing rate of the network and field $E(t)$) is periodic

Partial Synchronization - 2

Partial Synchronization in excitatory fully coupled networks (Phys. Rev. E 54, 5522 (1996))



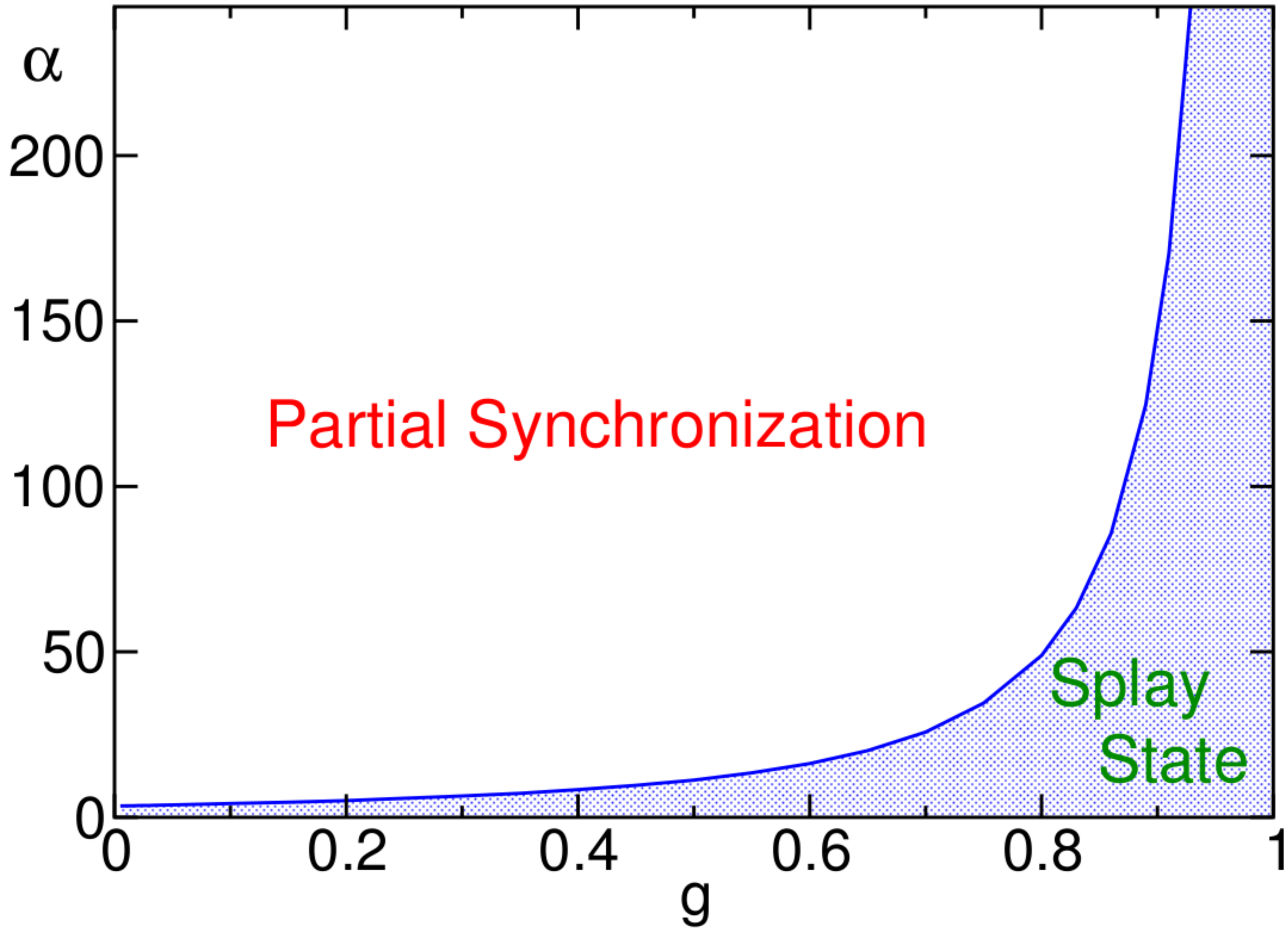


- supercritical Hopf bifurcation in the macroscopic variables (network rate and field $E(t)$)
- transition from synchronous to asynchronous state shown by the order parameter $\overline{m}$ [a]

[a] $m(t) = |1/N(\sum_k e^{2\pi i y_k})|$ where y_k is a phase-variable, function of the membrane potential v_k.

Partial Synchronization - 3

Partial Synchronization in excitatory fully coupled networks



Appendix: Model implementation - 1

Transformation of the set of differential equations into a discrete-time map

- STRATEGY $\rightarrow$ introducing auxiliary variables, the fields $E_i(t)$ resulting from the pulses received by each single neuron i

$$E_i(t) \equiv \frac{1}{N} \sum_{n|t_n<t} S_{i,l(n)} F(t-t_n) \qquad \longrightarrow \qquad \dot{v}_i = a - v_i + gE_i(t) \qquad i = 1, ..., N$$

- assuming that $F(t)$ is the Green's function of a linear differential equation of order d

$$E_i^{(d)} = \sum_{j=1,d} b_{i,j} E_i^{(j-1)} + \frac{k}{N} \sum_{n|t_n<t} S_{i,l(n)} \delta(t-t_n) \tag{5}$$

k is a suitable constant to normalize the area of the single shape

Appendix: Model implementation - 2

Transformation of the set of differential equations into a discrete-time map

- by integrating Eq. 5 from the initial condition $E_i^{(j)}(t_n) = E_i^{(j)}(n)$ (with $0 \leq j < d$), $v_i(t_n) \equiv v_i(n)$ for a time $\Delta t(n) = (t_{n+1} - t_n)$ between two consecutive spikes of the network

$$E_i(t_n + \Delta t) = E_i(n) + \sum_{j=1,d} c_{i,j} \mathrm{e}^{\lambda_j \Delta t}$$

  where λ_j's are the eigenvalues of the linear equation (for the sake of simplicity, we assume no degeneracy) and the coefficients $c_{i,j}$ are linear combinations of $E_i^{(j)}(n)$'s
- determining $\Delta t(n) = t_{n+1} - t_n$:

$$\Delta t = \min\{(\Delta t)_i : v_i(t_n + (\Delta t)_i) = \Theta\} \qquad i \in \{1, N\}$$

DEGREES OF FREEDOM: N(d+1) - 1